MATH 1700: SECTION 12.3: VECTORS

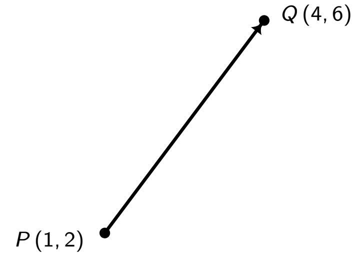
DEFINITION OF VECTOR:

A **vector** is a mathematical object which conveys two properties: a **magnitude** (or length) and a **direction**.

GEOMETRIC REPRESENTATION OF VECTORS:

A vector is represented geometrically as a directed line segment where the magnitude of the vector is taken to be the length of the line segment and the direction is made clear with the use of an arrow at one endpoint of the segment. When referring to vectors in this text, we shall adopt the 'arrow' notation, so the symbol $\vec{v}$ is read as 'the vector v'. Below is a typical vector $\vec{v}$ with endpoints $P\,(1,2)$ and $Q\,(4,6)$.

The point P is called the *initial point* or *tail* of $\vec{v}$ and the point Q is called the *terminal point* or *head* of $\vec{v}$. Since we can reconstruct $\vec{v}$ completely from P and Q, we write $\vec{v} = \overrightarrow{PQ}$, where the order of points P (initial point) and Q (terminal point) is important.

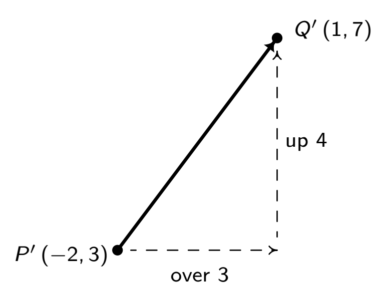


$\vec{v} = \overrightarrow{PQ}$

While it is true that P and Q completely determine $\vec{v}$, it is important to note that since vectors are defined in terms of their two characteristics, magnitude and direction, any directed line segment with the same length and direction as $\vec{v}$ is considered to be the same vector as $\vec{v}$, regardless of its initial point.

In the case of our vector $\vec{v}$ above, any vector which moves three units to the right and four up line segment containing $\vec{v}$ is $\frac{4}{3}$. from its initial point to arrive at its terminal point is considered the same vector as $\vec{v}$. The notation we use to capture this idea is the *component form* of the vector, $\vec{v} = \langle 3, 4\rangle$, where the first number, 3, is called the *x-component* of $\vec{v}$ and the second number, 4, is called the *y-component* of $\vec{v}$.

For example, if we wanted to reconstruct $\vec{v} = \langle 3, 4\rangle$ with initial point $P'(-2,3)$, then we would find the terminal point of $\vec{v}$ by adding 3 to the x-coordinate and adding 4 to the y-coordinate to obtain the terminal point $Q'(1,7)$:



$\vec{v} = \langle 3, 4\rangle$ with initial point $P'\,(-2,3)$.

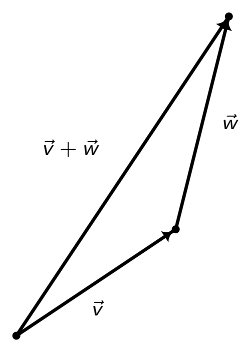
The component form of a vector is what ties these very geometric objects back to Algebra and ultimately Trigonometry. We generalize our example in our definition below.

COMPONENT FORM OF VECTORS: Suppose $\vec{v}$ is represented by a directed line segment with initial point $P(x_0, y_0)$ and terminal point $Q(x_1, y_1)$. The **component form** of $\vec{v}$ is given by

$$\vec{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle = \langle \Delta x, \Delta y \rangle$$

Using the language of components, we have that two vectors are equal if and only if their corresponding components are equal. That is, $\langle v_1, v_2 \rangle = \langle v_1', v_2' \rangle$ if and only if $v_1 = v_1'$ and $v_2 = v_2'$.

THE VECTOR SUM (RESULTANT) OF TWO VECTORS:



$\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$

EXAMPLE 1: A plane leaves an airport with an airspeed[1] of 175 miles per hour at a bearing of N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

[1]That is, the speed of the plane relative to the air around it. If there were no wind, plane's airspeed would be the same as its speed as observed from the ground. How does wind affect this?

VECTOR ADDITION (REPRISE): Suppose $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$. The vector $\vec{v} + \vec{w}$ is defined by

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$

EXAMPLE 2: $\vec{v} = \langle 3, 4 \rangle$ and suppose $\vec{w} = \overrightarrow{PQ}$ where $P(-3, 7)$ and $Q(-2, 5)$.

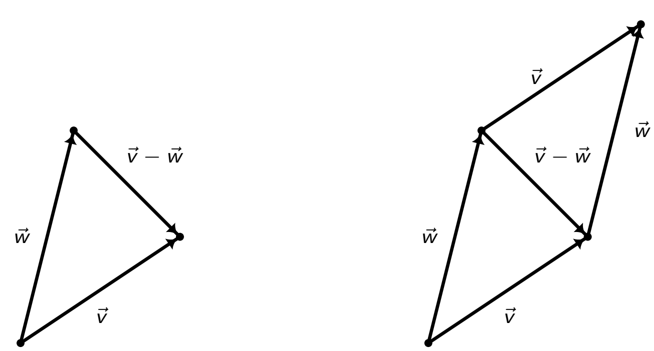
Find $\vec{v} + \vec{w}$ and interpret this sum geometrically.

THE ZERO VECTOR: The zero vector is defined as: $\vec{0} = \langle 0, 0 \rangle$.

PROPERTIES OF VECTOR ADDITION:

- **COMMUTATIVE:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
- **ASSOCIATIVE:** For all vectors $\vec{u}, \vec{v}$ and $\vec{w}$, $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
- **IDENTITY:** For all vectors $\vec{v}$,
$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}.$$
The vector $\vec{0}$ acts as the additive identity for vector addition.
- **INVERSE:** For every vector $\vec{v} = \langle v_1, v_2 \rangle$, the vector $\vec{u} = \langle -v_1, -v_2 \rangle$ satisfies
$$\vec{v} + \vec{u} = \vec{u} + \vec{v} = \vec{0}.$$
That is, the additive inverse of a vector is the vector of the additive inverses of its components..

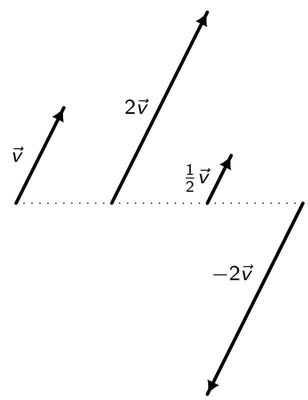
The proofs of these properties are detailed in the text and can be verified algebraically and geometrically. Note that the inverse property allows us to define vector subtraction as follows: the vector $\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$ where the vector '$-\vec{w}$' is the additive inverse of $\vec{w}$. Geometrically:



SCALAR MULTIPLICATION: If k is a real number and $\vec{v} = \langle v_1, v_2 \rangle$, we define $k\vec{v}$ by

$$k\vec{v} = k \langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle$$

Scalar multiplication by k in vectors can be understood geometrically as scaling the vector (if $k > 0$) or scaling the vector and reversing its direction (if $k < 0$) as demonstrated below.



Note that $(-1)\vec{v} = (-1)\langle v_1, v_2 \rangle = \langle (-1)v_1, (-1)v_2 \rangle = \langle -v_1, -v_2 \rangle = -\vec{v}$, which is what we would expect. This and other properties of scalar multiplication are summarized in the theorem below.

PROPERTIES OF SCALAR MULTIPLICATION:

- **ASSOCIATIVE:** For every vector $\vec{v}$ and scalars k and r, $(kr)\vec{v} = k(r\vec{v})$.
- **IDENTITY:** For all vectors $\vec{v}$, $1\vec{v} = \vec{v}$.
- **ADDITIVE INVERSE:** For all vectors $\vec{v}$, $-\vec{v} = (-1)\vec{v}$.
- **DISTRIBUTIVE PROPERTY OF SCALAR MULTIPLICATION OVER SCALAR ADDITION:** For every vector $\vec{v}$ and scalars k and r,

$$(k + r)\vec{v} = k\vec{v} + r\vec{v}$$

- **DISTRIBUTIVE PROPERTY OF SCALAR MULTIPLICATION OVER VECTOR ADDITION:** For all vectors $\vec{v}$ and $\vec{w}$ and scalars k,

$$k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$$

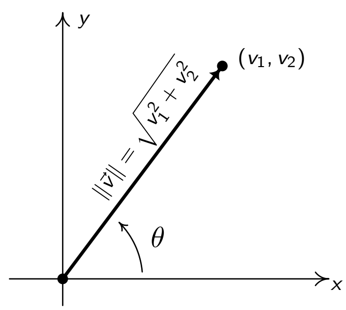
- **ZERO PRODUCT PROPERTY:** If $\vec{v}$ is vector and k is a scalar, then

$$k\vec{v} = \vec{0} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad \vec{v} = \vec{0}$$

The proofs of the above properties ultimately boils down to the definition of scalar multiplication and properties of real numbers and are detailed in the text.

EXAMPLE 3: Solve $5\vec{v} - 2\left(\vec{v} + \langle 1, -2\rangle\right) = \vec{0}$ for $\vec{v}$.

A vector whose initial point is $(0, 0)$ is said to be in **standard position**. If $\vec{v} = \langle v_1, v_2\rangle$ is plotted in standard position, then its terminal point is necessarily (v_1, v_2).



$\vec{v} = \langle v_1, v_2\rangle$ in standard position.

Recall the magnitude of vector $\vec{v}$ is the length of the directed line segment representing $\vec{v}$. When plotted in standard position, the length of this line segment is none other than the distance from the origin $(0, 0)$ to the point (v_1, v_2). Hence, the magnitude of $\vec{v}$, which we denote $\|\vec{v}\|$, is given by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$.

Turning to the notion of direction, we note that the point (v_1, v_2) is on the terminal side of the angle θ depicted in the diagram above. Hence, $v_1 = \|\vec{v}\|\cos(\theta)$ and $v_2 = \|\vec{v}\|\sin(\theta)$. Therefore:

$$\begin{aligned} \vec{v} &= \langle v_1, v_2\rangle \\ &= \langle \|\vec{v}\|\cos(\theta), \|\vec{v}\|\sin(\theta)\rangle \\ &= \|\vec{v}\|\,\langle \cos(\theta), \sin(\theta)\rangle \end{aligned}$$

MAGNITUDE AND DIRECTION:

Suppose $\vec{v}$ is a vector with component form $\vec{v} = \langle v_1, v_2\rangle$. Let θ be an angle in standard position whose terminal side contains the point (v_1, v_2).

- The **magnitude** of $\vec{v}$, denoted $\|\vec{v}\|$, is given by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$
- If $\vec{v} \neq \vec{0}$, the **(vector) direction** of $\vec{v}$, denoted $\hat{v}$ is given by $\hat{v} = \langle \cos(\theta), \sin(\theta)\rangle$

Taken together, we get $\vec{v} = \langle \|\vec{v}\|\cos(\theta), \|\vec{v}\|\sin(\theta)\rangle = \|\vec{v}\|\,\hat{v}$.

PROPERTIES OF MAGNITUDE AND DIRECTION: Suppose $\vec{v}$ is a vector.

- $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$
- For all scalars k, $\|k\,\vec{v}\| = |k|\|\vec{v}\|$.
- If $\vec{v} \neq \vec{0}$ then $\vec{v} = \|\vec{v}\|\hat{v}$, so that $\hat{v} = \left(\dfrac{1}{\|\vec{v}\|}\right)\vec{v}$.

The proofs of these properties are detailed in the text.

EXAMPLE 4:

1. Find the component form of the vector $\vec{v}$ with $\|\vec{v}\| = 5$ so that when $\vec{v}$ is plotted in standard position, it lies in Quadrant II and makes a 60° angle with the negative x-axis.

2. For $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, find $\|\vec{v}\|$ and θ, $0 \leq \theta < 2\pi$ so that $\vec{v} = \|\vec{v}\| \left\langle \cos(\theta), \sin(\theta) \right\rangle$.

3. For the vectors $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$, find the following.

   (a) $\hat{v}$

   (b) $\|\vec{v}\| - 2\|\vec{w}\|$

   (c) $\|\vec{v} - 2\vec{w}\|$

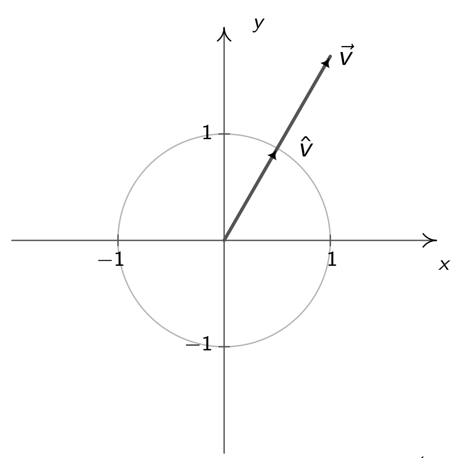
   (d) $\|\hat{w}\|$

EXAMPLE 5: A plane leaves an airport with an airspeed of 175 miles per hour with bearing N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

UNIT VECTORS: Let $\vec{v}$ be a vector. If $\|\vec{v}\| = 1$, we say that $\vec{v}$ is a **unit vector**.

Note that if $\vec{v}$ is a unit vector, then necessarily, $\vec{v} = \|\vec{v}\|\hat{v} = 1 \cdot \hat{v} = \hat{v}$. Conversely if $\|\hat{v}\| = 1$, then $\hat{v}$ is a unit vector. In other words, unit vectors are direction vectors and vice-versa. Indeed, the vector $\hat{v}$ which we have defined as 'the *direction* of $\vec{v}$' is often described as 'the *unit vector in the direction* of $\vec{v}$.' If $\vec{v}$ is a unit vector we write '$\hat{v}$' as opposed to '$\vec{v}$' because we reserve the '$\hat{\ }$' notation for unit vectors. The process of multiplying a nonzero vector by the factor $\frac{1}{\|\vec{v}\|}$ to produce a unit vector is called '**normalizing** the vector.'

The terminal points of unit vectors, when plotted in standard position, lie on the Unit Circle. As a result, we visualize normalizing a nonzero vector $\vec{v}$ as shrinking[2] its terminal point, when plotted in standard position, back to the Unit Circle.



Visualizing vector normalization $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right)\vec{v}$

[2] ... if $\|\vec{v}\| > 1$ ...

PRINCIPAL UNIT VECTORS IN THE PLANE:

- The vector $\hat{\imath}$ is defined by $\hat{\imath} = \langle 1, 0 \rangle$
- The vector $\hat{\jmath}$ is defined by $\hat{\jmath} = \langle 0, 1 \rangle$

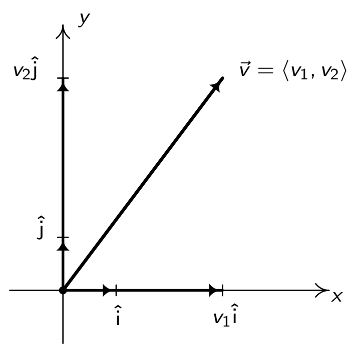
Geometrically, in the xy-plane, the vector $\hat{\imath}$ as represents the positive x-direction, whereas the vector $\hat{\jmath}$ represents the positive y-direction. We have the following 'decomposition' theorem:

PRINCIPAL VECTOR DECOMPOSITION THEOREM:
Let $\vec{v}$ be a vector with component form $\vec{v} = \langle v_1, v_2 \rangle$. Then $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$.

Since $\hat{\imath} = \langle 1, 0 \rangle$ and $\hat{\jmath} = \langle 0, 1 \rangle$, we have from the definition of scalar multiplication and vector addition that

$$v_1\hat{\imath} + v_2\hat{\jmath} = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = \langle v_1, v_2 \rangle = \vec{v}$$

Geometrically, the situation looks like this:



$$\vec{v} = \langle v_1, v_2 \rangle = v_1\hat{\imath} + v_2\hat{\jmath}.$$

EXAMPLE 6: A 50 pound speaker is suspended from the ceiling by two supports. If one of them makes a 60° angle with the ceiling and the other makes a 30° angle with the ceiling, what is the tension on each support?

HOMEWORK: Section 12.3: 1064: 1-61 every other odd.